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THE MATHEMATICAL PRINCIPLES OF PICTORIAL REPRESENTATION

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THE object of artists in painting pictures, in making drawings and designs, in creating works of plastic art and sculpture, is to represent a landscape with the moods of nature, human and other living beings in their association with historic and social events, to portray and sculpture single figures and groups of figures of importance in the cultural development of a people; or merely to depict incidents of human interest in everyday life. The purpose of decorative art is embellishment and the breaking of the monotony of blank surfaces where purely utilitarian considerations do not interfere or make decoration desirable. Contrary to popular opinion that fine arts are or should be concerned with the representation of the beautiful only, they sometimes intend to create a certain impression, or to influence public opinion for a certain purpose which is moral rather than specifically beautiful. By his incomparable battle-scenes the great Russian painter Wereschtschagin, who went down with the *Petropawlowsk* near Port Arthur during the Russian-Japanese war, for example, wanted to impress the people with the horrors of war.

The methods and styles by which artists accomplish their purpose are of such a great variety that only a complete history of fine arts could convey an idea of what has been accomplished in this ideal endeavor of the human mind.

In sculpture the form of the represented model is mostly similar, if not identical, with that of the true object, while in painting a plane surface must serve as a base of representation, so that, in general, there is no geometrical similitude between the external forms and their plane pictures. The same is true of relief-modelling in which the relation between the plastic and the original figure in space is that of a certain variable scale. The laws which govern these changes in the correspondence of forms are expressed by the technical term collineation, or rather by a particular kind of collineation, called perspective. We shall be concerned with this alone, *i. e.*, with the geometrical laws which connect the original form with its pictorial representation. Other elements of depiction, like color, shades and shadows, aerial perspective, and other effectives, lie beyond the scope of this essay. I shall first explain the principles upon which artistic perspective is based. These are so simple that any person with elementary geometrical knowledge is able to understand them. A certain minimum of theoretical considerations of this kind is necessary for the comprehension of the critical and historical remarks on the application of perspective in painting during various historic periods.

The fundamental idea of representation, or depiction, is contained in the principle of correspondence between the elements of space, *i. e.*, points, lines, and planes of the object and the elements of the figure which is supposed to represent the same. This figure we shall simply call the perspective (relief) of the object. This is merely a special concrete case of the theory of correspondences, a theory which is of the greatest importance in many lines of modern mathematical research.

In Fig. 1, let p' be a fixed plane, usually in a vertical position, and C a fixed point in space, not located in p', called picture-plane. The point C is called center of perspective and takes the place of the eye of the observer. We may think of p' as the film of a camera, or as the plane of the canvas upon which the picture is to be painted. To obtain the perspective of any point A in space, join A to C. The intersection A' of AC with p' is then the perspective of A. Thus, to every point in space corresponds a point in p'. To a point P which lies in

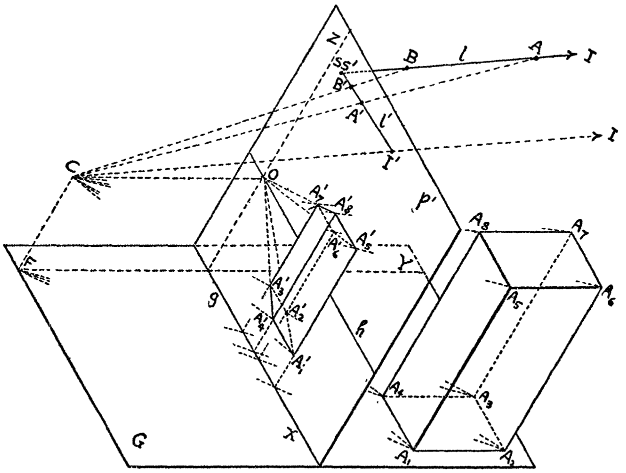

FIG. 1.

p' itself corresponds the point itself, *i. e.*, P' coincides with P. A straight line, or simply a line l is determined by two points, say A and B, so that its perspective is obtained as the line l' joining the perspectives A' and B' of A and B. The perspective l' of l is also obtained as the intersection of the plane passing through C and l with p'. A line, like CA, is called the projecting ray of A. When a point I moves on a line l beyond any finite region, then the projecting ray CI becomes parallel to l, and the perspective of I, *i. e.*, the intersection I' of CI with p', is called vanishing point of the line l. If S is the point where l pierces the plane p', then S' coincides with S, so that l' may also be obtained as the line which joins S' and I'. From the definition of the vanishing point of a line it follows that parallel lines in space have the same vanishing point, so that the perspectives of parallel lines all converge towards the same point, as is well known. The improper position of the point I on l beyond any finite point of l is in projective geometry defined as the infinite point of l. It is likewise possible to assume without meeting contradictions that the non-finite region of a plane is a straight line, that of ordinary space a plane. In this manner all statements in the foregoing correspondence become perfectly general. Parallel lines are now lines through an infinite point, to which corresponds the vanishing point of those lines. Lines parallel to a given plane pass through the infinite line of this plane, so that their vanishing points lie on a line in p', which corresponds to the infinite line of the given plane, and which is obtained as the line of intersection of the plane through C, parallel to the given plane, with the picture-plane p'. The only parallel lines which are projected into parallel lines are parallel lines which themselves are parallel to p'. If the picture-plane stands vertically upon a horizontal ground-plane G which intersects the picture-plane in the ground line g, then the vanishing points of all horizontal lines lie on a vanishing line h called "horizon."

In the terminology of artistic perspective these results may be stated as follows:

1. Corresponding points of the object and of its perspective lie on rays through the center.
2. Corresponding lines meet in points of the picture-plane. The perspective of a line through the center is the point of intersection of the line with the picture-plane.
3. Parallel lines have the same vanishing point through which their perspectives pass.

4. The vanishing points of lines parallel to the same plane lie on the same line.

5. The vanishing points of horizontal lines lie on the horizon.

6. Parallel lines parallel to the picture-plane are projected into parallel lines.

7. Vertical lines are projected into vertical lines.

8. The point of intersection of two lines is projected into the point of intersection of the perspectives of those lines.

9. Plane figures parallel to the picture-plane are projected into similar figures.

10. The diagonals of an ordinary quadrangle, which is the perspective of a rectangle (square), intersect in a point which is the perspective of the center of the rectangle (square).

11. The vanishing point of lines perpendicular to the picture-plane is called the eye-point of the perspective. It is also the foot-point of the perpendicular from the center to the picture-plane.

12. The vanishing points of two perpendicular lines lie on a semicircle through C over the segment between the vanishing points as a diameter.[1]

These rules, which might be increased, are sufficient to test the correctness of the perspective of a painting.

In Fig. 1, the eye-point is denoted by O. The rectangular prism $A_1A_2A_3A_4A_5A_6A_7A_8$ is placed upon the ground-plane G and with two faces $A_1A_5A_8A_4$ and $A_2A_6A_7A_3$ parallel to the picture-plane p'. Accordingly, $A_1'A_4'$, $A_5'A_8'$, $A_2'A_3'$, $A_6'A_7$, are parallel to g; $A_1'A_5'$, $A_2'A_6'$, $A_3'A_7'$, $A_4'A_8'$ are vertical and the rest of the edges in the perspective, $A_1'A_2'$, $A_4'A_3'$, $A_5'A_6'$, $A_8'A_7'$, prolonged, pass through O on the horizon h.

Fig. 2 shows the application of the principles of perspective to landscape drawing. The perspectives of parallel horizontal lines, when produced, meet in some point of the horizon h, while vertical lines appear as vertical lines. In the landscape the background is formed by gently rolling hills. The horizon will therefore be slightly below the crest of the hills. Also the ruts

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[1] For those readers who are familiar with the elements of analytic geometry, the analytic form of the perspective may be established as follows: Let G be the xy-plane, p' the xz-plane, and the plane through C perpendicular to p' and G the yz-plane, and the space in which the prism stands, the octant in which all coordinates are positive. Then when $FC = c$, $CO = b$, the coordinates x, y, z of a point A and the coordinates x', z' of the corresponding point A' are related by the formulas

$$x' = \frac{bx}{y+b}, \qquad z' = \frac{cy+bz}{y+b}.$$

of the level wagon road, which run along more or less parallel lines, when extended will meet in a point in or near the horizon. Notice also the proper perspective reduction of the scale of distant buildings and trees.

In a similar manner we may establish the geometrical principles of theatrical and relief perspective. Such a correspondence between the elements of the object and those of the relief is technically known as a perspective collineation of space; and is determined when the center C, the plane of perspective (axial plane) s, and two corresponding points A and A' on a ray through C are given. If any other point B is given, the corresponding point B' is found by prolonging the line joining A and B to the intersection S with the axial plane s, Fig. 3. Then, join S to A', and find the intersection B' of SA' with CB prolonged, if necessary. B' is the required point. Consider next the indefinitely extended lines l through A and B, and l' through A' and B'. The infinite point I of l' is projected into a point I' of l'. Conversely, there is a point J on l whose perspective J' is the infinite point of l'. In order to shorten the theoretical discussion it is sufficient for our purpose to state that if the points I' and J are constructed for every pair of corresponding lines l and l', it is found that all points I' lie in a plane q', all points J in a plane r, both of which are parallel to the axial plane s. Again we may set up a number of rules:

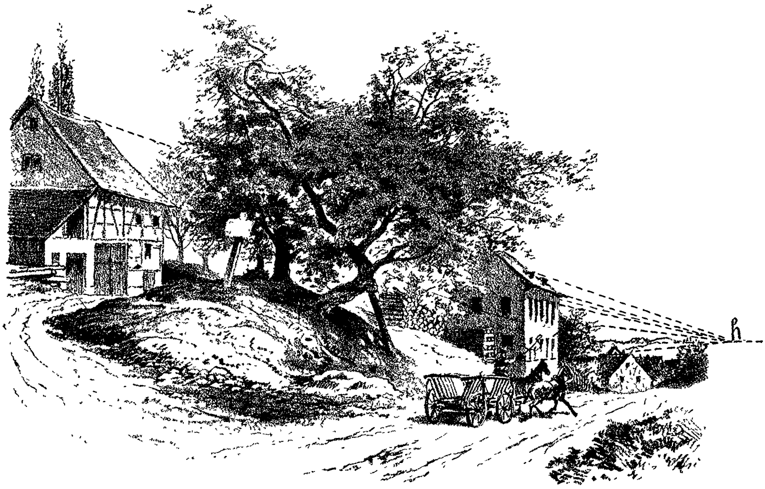

FIG. 2.

1. Corresponding points lie on rays through the center C.
2. Corresponding lines meet in points of the axial plane s.

3. The perspectives (reliefs) of parallel lines meet in a point, those of parallel planes in a line of the vanishing plane q'.

4. The vanishing points of horizontal lines lie on the horizon h, which is in q'.

5. When the axial plane s (hence also q' and r) stands vertically on a ground-plane G, cutting the latter in the ground-line g, the horizon is parallel to g, and to vertical lines correspond vertical lines.

6. Corresponding planes meet in a line of s.

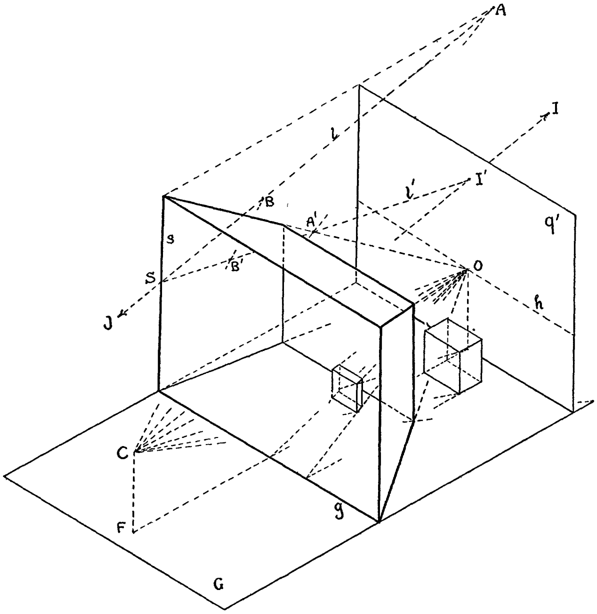

Fig. 3.

7. To planes parallel to s correspond planes parallel to s.

8. To lines perpendicular to s correspond lines passing through the so-called eye-point O of q'.

This list is, of course, by no means complete, and might be indefinitely extended.[2]

[2] To obtain the analytical form of theatrical perspective choose again G as the xy-plane, s as the xz-plane, and the plane through C perpendicular to s as the yz-plane. Let b be the distance of C from s, the space in which the object is located as the octant in which all coordinates are positive, e the distance of r from C, e also the distance from s to q', x, y, z the coordinates of a point A, x', y', z' the coordinates of the corresponding point A', then the coordinates are related by

$$x' = \frac{(b+e)x}{y+b+e}, \qquad y' = \frac{ey}{y+b+e}, \qquad z' = \frac{cy+(b+e)z}{y+b+e}.$$

On the stage of a theater the axial plane s may be taken as the plane of the front-curtain, the plane q' as the rear wall of the stage. The observer is supposed to take the place of C. Naturally this theoretically ideal condition cannot be realized practically. There are many observing centers, so that considerable latitude and deviation from the mathematical theory is admissible. Although the mathematician speaks of the plane of the curtain as the invariant plane of the collineation, it is of course not his intention to impose his theory upon the theatergoer.

Fig. 4 represents a typical example of theatrical perspective.

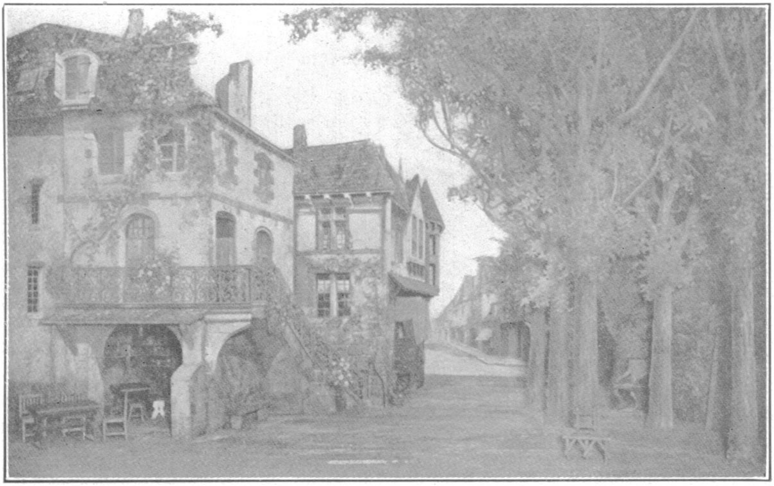

FIG. 4. STAGE-SETTING FOR ACT I. OF PUCCINI'S MANON LESCAUT AT THE METROPOLITAN IN NEW YORK.

It shows the setting of Act I. of the opera "Manon Lescaut" by Puccini at the Metropolitan in New York.

Fig. 5, likewise, may give an idea of the wonderful perspective effect of the stage-setting of the ballroom in the Duke's Palace at a performance of Verdi's Rigoletto, also at the Metropolitan.

The development of the principles of perspective in their practical applications to painting has a long history. It evolved from the primitive, purely intuitional efforts of ancient artists, and reached the present state of a perfect system only after many centuries of improvements and rational scientific coordination with geometry. As a matter of fact, synthetic, or projective geometry grew out of geometrical discoveries revealed by the study of artistic perspective. Desargues's (1593–1662) brilliant accomplishments in this field make him one of the foremost pioneers of modern geometry. On the other hand, projective geometry had a great influence upon artistic perspective and the comprehension of its essential simplicity. We have here again a characteristic example for the interesting fact that a branch of mathematical science grew out of a field of practical, more or less intuitional rules, which in its turn was used to clarify the original primitive notions, to simplify and systematize the collection of practical rules and to make their application easier.

Traces of pictorial representation may be found almost as far back as we have knowledge of the prehistoric human race. It is, however, not until possibly thousands of centuries later that we see Babylonians, Egyptians, Assyrians and other races make extensive use of graphic representations in their cultural development. But they do not have any knowledge of how to make more than two dimensions appear on a picture. All objects are shown in elevation, or in a front view. When a plan, or top view, is drawn, vertical objects are depicted on the plan as they would appear in elevation.

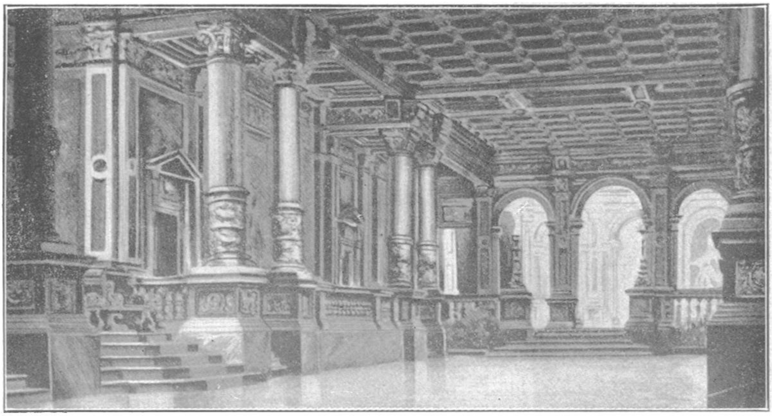

FIG. 5. SETTING OF ACT I. IN VERDI'S RIGOLETTO AT THE METROPOLITAN.

The ancient Greeks, although even now unsurpassed in sculpture and in the true conception of architectural laws, were not much better when it came to drawing and painting. Front-views were the customary methods of decoration and drawing. As there seem to be no paintings of Greek antiquity in existence, it is impossible to arrive at a correct evaluation of their treatment of form. From writings on objects of art it would appear that the Greeks were not in possession of a deductive system, and that they relied entirely upon empiric, *i. e.*, in this case.

visual impressions. On the other hand it is surprising what anecdotes tell us of some marvelous illusions which Greek paintings seem to have produced. Thus the "grapes of Zeuxis" were painted in such true imitation of nature that the birds tried to pick them. Parrhasius painted an open door with curtain drapery on a wall, through which people, who did not know of the illusion, tried to pass.

As appears from mural decorations of Pompey, Roman artists had some crude knowledge of empirical perspective. But it was without control by correctly deduced geometrical laws, and therefore faulty.

The same may be said of the early Italian painters, like Giotto (1266–1337) and his pupils, whose frescoes in the chapels of St. Croce in Florence, of Madonna dell' Arena in Padua, and others, exhibit very little knowledge of perspective. It was not until the early Renaissance, the Quartocento, that perspective was established as a science. During this period we find writings on the principles of perspective by the architect Brunellesco (1377–1466), by the sculptor Donatello (1386–1468), and by the architect Alberti (1404–1472). The full development and mastery of perspective for artistic purposes was accomplished by the great masters of the Renaissance at the end of the fifteenth and in the sixteenth century: Leonardo da Vinci (1452–1519), Raffael Santi (1483–1520), and Michel Angelo Buonarotti (1475–1564). Leonardo da Vinci, a universal genius, who attained fame not only as a painter, but also in engineering, in physics, in anatomy, and in music, wrote a Trattato della pittura, which contained also a monograph on perspective. Unfortunately this important work has been lost.

As an example of the complete mastery of the laws of perspective I refer to Raffael's "The Wedding of Maria" in the Milan gallery, of which Fig. 6 is a half-tone reproduction. In the whole representation of the architectural features the perspective correspondence between the external world and the canvas is correctly and minutely established. Also the human figures are portrayed with equal carefulness and proper reductions of the scale. In spite of the geometrical correctness of the construction no unpleasing distortions of form appear in the picture. This is the more remarkable as at that time no control by photographic processes was known.

Many other paintings of equal renown by Raffael and other painters of this period might be analyzed, which would merely corroborate the statement concerning the highly and rationally developed sense of form perception of these masters.

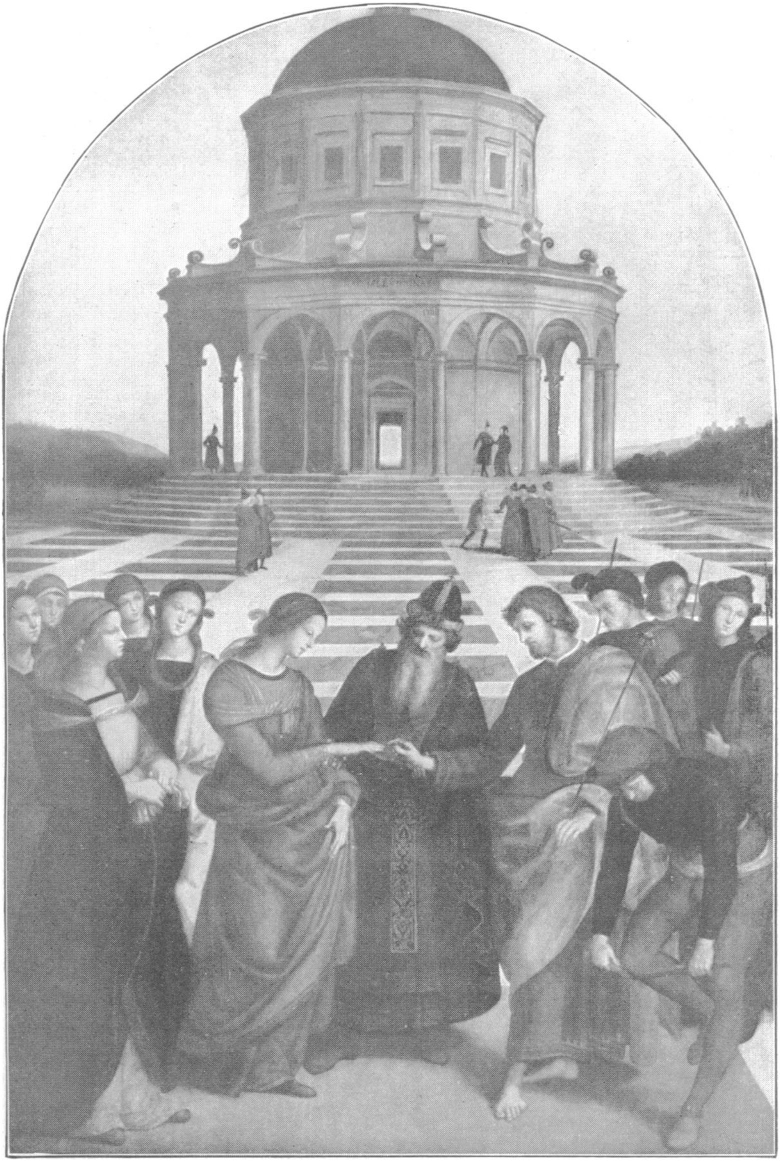

FIG. 6. RAFFAEL, THE WEDDING OF MARIA.

It is strange that during the following period some artists became careless in regard to the coordination of their painting with consistent perspective forms. Paolo Veronese's (1528–1588) "The Wedding of Cana," which is in the Louvre (Paris) gallery, for example, contains seven eye-points and five horizons. This is unfortunate, as the painting is otherwise of great artistic value. Fr. Bossuet showed in his "Traité de perspective" (Brussels, 1871), by a reconstruction of the architectural features of Veronese's painting with one eye-point, that the intended impression and effect of the painting do not suffer in the least from a correct construction.

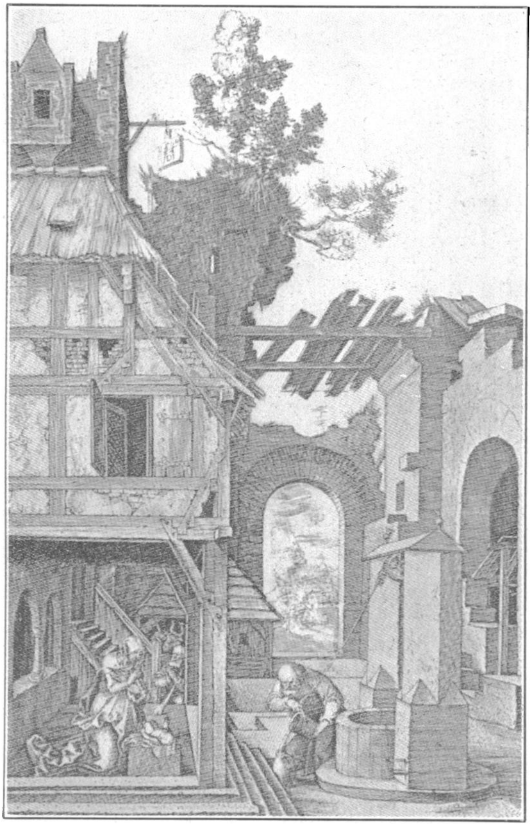

FIG. 7. ALBRECHT DÜRER, BIRTH OF CHRIST.

Of other than Italian painters of the Renaissance I shall simply mention Albrecht Dürer (1471–1528), who, like Raffael and his compatriots, had a perfect knowledge of the principles of perspective, as revealed by all his paintings. He wrote the first German work on perspective, called "Unterweisung der

Messung mit Zirkel und Richtscheyt," in which a number of devices (glass plate) are explained how to make true perspective constructions. The "Birth of Christ," reproduced in Fig. 7, is an example of Dürer's art with its careful attention to perspective details.

Most of the great painters of modern times show a profound knowledge of perspective, where proper delineation and reduction of scales are imperative. This, of course, is always the case when architectural features form a part of the setting. See, for example, some of Boecklin's famous paintings, or Rochegrosse's "The Flagellants," in collections of reproductions of modern paintings. Certain styles and varieties of painting require less or no geometrical preliminary work. It must be said, however, that in certain cases deviations from the geometrical laws are permissible. But great painters are conscious of such deviations and do not introduce them deliberately, when not needed. The mistakes of some mediocre painters and their flagrant neglect of correct forms are due not so much to conscious modifications as to their ignorance of the laws of perspective. The mastery of these laws requires some serious thinking and study, which is too arduous a task for so many "heroes of the brush." In so many cases of modernism and impressionism the claim of originality and progressiveness is merely a flimsy cover or excuse for the painter's inability and his ignorance or misunderstanding of some of the most fundamental principles of fine arts.

A thorough mastery of these requires years of intensive study and practise. The lack of these is responsible for much of the mediocrity of the present day.